

# DIFFERENTIAL EQUATIONS FOR SINGULAR VALUES OF PRODUCTS OF GINIBRE RANDOM MATRICES

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**ABSTRACT.** It was proved by Akemann, Ipsen and Kieburg [6] that squared singular values of products of  $M$  complex Ginibre random matrices form a determinantal point process whose correlation kernel is expressible in terms of Meijer's  $G$ -functions. Kuijlaars and Zhang [22] recently showed that at the edge of the spectrum, this correlation kernel has a remarkable scaling limit  $K_M(x, y)$  which can be understood as a generalization of the classical Bessel kernel of Random Matrix Theory. In this paper we investigate the Fredholm determinant of the operator with the kernel  $K_M(x, y)\chi_J(y)$ , where  $J$  is a disjoint union of intervals,  $J = \cup_j (a_{2j-1}, a_{2j})$ , and  $\chi_J$  is the characteristic function of the set  $J$ . This Fredholm determinant is equal to the probability that  $J$  contains no particles of the limiting determinantal point process defined by  $K_M(x, y)$  (the gap probability). We derive Hamiltonian differential equations associated with the corresponding Fredholm determinant, and interpret them as the monodromy preserving deformation equations of the Jimbo, Miwa, Mōri, Ueno and Sato theory. In the special case  $J = (0, s)$  we give a formula for the gap probability in terms of a solution of a system of non-linear ordinary differential equations.

## 1. INTRODUCTION

It is a well-known fact that a natural language for a description of probabilistic quantities of interest in the theory of exactly solvable random matrix models is the language of non-linear and partial differential equations. We refer the reader to the works of Tracy and Widom [30]-[33], Adler, Shiota and van Moerbeke [2, 3], to the surveys by van Moerbeke [24, 25], and to the book by Forrester [12] for an introduction to this aspect of Random Matrix Theory, and for main results in this area of research.

One classical example of such a description is that of singular values of a random complex Ginibre matrix. Alternatively, one can think about eigenvalues of a complex Wishart matrix, i.e. about eigenvalues of a matrix  $X^*X$ , where  $X$  is a random complex Ginibre matrix of size  $N \times K$ . By definition, a random complex Ginibre matrix is a rectangular matrix whose entries are independent standard complex Gaussian variables. It turns out that the squared singular values of a complex Ginibre matrix form a determinantal process on  $(0, \infty)$  (called the classical Laguerre ensemble). A remarkable feature of this determinantal point process is that its correlation kernel can be written explicitly in terms of the classical Laguerre polynomials. This allows to study different asymptotic regimes of the classical Laguerre ensemble, see, for example, Forrester [12], Chapter 7. When one rescales the classical Laguerre ensemble at the hard edge of the spectrum the limiting determinantal point process with the kernel

$$(1.1) \quad K_{\text{Bessel}}(x, y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J'_\nu(\sqrt{y}) - \sqrt{x}J'_\nu(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}$$

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arises. (Here  $J_\nu(z)$  stands for the Bessel function of order  $\nu$ , and  $\nu = N - K$ ). Moreover, different probabilistic quantities of interest are expressible in terms of Fredholm determinants of operators defined by this kernel. For example, the Fredholm determinant of the operator with the kernel  $K_{\text{Bessel}}(x, y)\chi_J(y)$ , where  $J$  is a disjoint union of intervals,  $J = \cup_j(a_{2j-1}, a_{2j})$ , and  $\chi_J$  is the characteristic function of the set  $J$  can be understood as the gap probability. This is the probability of the event that there are no particles of the limiting determinantal process in  $J$ . The theory of the Fredholm determinant defined by the Bessel kernel is developed by Tracy and Widom [32]. Namely, Tracy and Widom [32] obtain a system of partial differential equations for the logarithmic differential of this Fredholm determinant. These partial differential equations admit a Hamiltonian formulation in which the end points  $a_j$  of the intervals play the role of multi-time variables. Subsequently, it was shown in Palmer [29], Harnad, Tracy, and Widom [15], Harnad [14] that the partial differential equations characterizing the Fredholm determinant of the operator with the kernel  $K_{\text{Bessel}}(x, y)\chi_J(y)$  can be viewed as a special case of the monodromy preserving deformation equations of the Jimbo, Miwa, Mōri, Ueno and Sato theory [19]-[21]. In this framework, the Fredholm determinant of the operator defined by the Bessel kernel can be understood as the tau-function, and the analysis of the Fredholm determinant can be considered as a special case of the analysis of tau-functions.

In the case of a single interval Tracy and Widom [32] give a representation of the Fredholm determinant in terms of a solution of the Painlevé V equation. This allows to understand the asymptotic behavior of the probability of the event that there is no particle of the limiting determinantal point process in  $(0, s)$ , as  $s \rightarrow \infty$ .

It is the aim of the present paper to extend some of the results mentioned above to *products of independent Ginibre matrices*. Such products arise in very different areas of research, see, for example, Müller [26, 27], Akemann, Ipsen, and Kieburg [6] for applications in the theory of telecommunications. In the context of this paper the most important fact is that products of independent Ginibre matrices lead to determinantal point processes both in the complex plane  $\mathbb{C}$  and on the real line  $\mathbb{R}$ . This has been shown recently by Akemann and Burda [4], Akemann, Kieburg and Wei [5], Akemann, Ipsen and Kieburg [6] (see also Adhikari, Reddy, Reddy, and Saha [1]). The correlation kernels of such determinantal processes can be expressed in terms of Meijer's  $G$ -functions, which enables to investigate the statistics of eigenvalues and of singular values for products of independent Ginibre matrices by the usual methods of Random Matrix Theory. We refer the reader to papers by Akemann and Strahov [7], Zhang [35], Kuijlaars and Zhang [22] (in addition to papers just mentioned above) for some recent results in this direction. In particular, it was proved by Akemann, Ipsen and Kieburg [6] that squared singular values of products of  $M$  complex Ginibre random matrices form a determinantal point process on  $\mathbb{R}_{>0}$  whose correlation kernel is expressible in terms of Meijer's  $G$ -functions. Kuijlaars and Zhang [22] show that at the edge of the spectrum, this correlation kernel has a remarkable scaling limit  $K_M(x, y)$  which can be understood as a generalization of the classical Bessel kernel of Random Matrix Theory. In this paper we investigate the Fredholm determinant of the operator with the kernel  $K_M(x, y)\chi_J(y)$ , where  $J$  is a disjoint union of intervals,  $J = \cup_j(a_{2j-1}, a_{2j})$ , and  $\chi_J$  is the characteristic function of the set  $J$ . We derive Hamiltonian differential equations associated with the corresponding Fredholm determinant, and interpret them as the monodromy preserving deformation equations of the Jimbo, Miwa, Mōri, Ueno and Sato [19]-[21] theory. In the special case  $J = (0, s)$  we give a formula for the gap probability in terms of a solution of a system of non-linear ordinary differential equations.

This paper is organized as follows. In Section 2 we summarize exact and asymptotic results on singular values of products of Ginibre random matrices which are relevant for this work. In particular, in Section 2 we present an explicit formula for the limiting correlation kernel  $K_M(x, y)$  found in Kuijlaars and Zhang [22]. In Section 3 we state new results obtained in this paper. Proposition 3.3 gives a system of partial differential equations associated with the Fredholm determinant of the operator with the kernel  $K_M(x, y)\chi_J(y)$ . Proposition 3.4 provides a Hamiltonian formulation of these partial differential equations, and in Section 3.4 we interpret them as isomonodromic deformation equations, see Proposition 3.6 and Proposition 3.7. Finally, in Section 3.5 the special case  $J = (0, s)$  is considered, and Proposition 3.9 gives a formula for the probability that no particles of the determinantal process lie in the interval  $(0, s)$ . Section 4 is devoted to the derivations of differential equations and to proofs of other results stated in Section 3.

## 2. SINGULAR VALUES OF PRODUCTS OF RANDOM COMPLEX MATRICES

To present the results on singular values of products of random complex matrices let us adopt the same notation and definitions for Meijer's  $G$ -function as in Luke [23], Section 5.2. Namely, the Meijer  $G$ -function  $G_{p,q}^{m,n} \left( x \left| \begin{smallmatrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{smallmatrix} \right. \right)$  is defined as

$$G_{p,q}^{m,n} \left( z \left| \begin{smallmatrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{smallmatrix} \right. \right) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds.$$

Here an empty product is interpreted as unity,  $0 \leq m \leq q$ ,  $0 \leq n \leq p$ , and the parameters  $\{a_k\}$  ( $k = 1, \dots, p$ ) and  $\{b_j\}$  ( $j = 1, \dots, m$ ) are such that no pole of  $\Gamma(b_j - s)$  coincides with any pole of  $\Gamma(1 - a_k + s)$ , and  $C$  is an appropriate integration contour. We assume that  $z \in \mathbb{C} \setminus \{0\}$ . If  $p = 0$ , then  $n = 0$ , and we write the corresponding Meijer  $G$ -function as  $G_{0,q}^{m,0} \left( x \left| \begin{smallmatrix} b_1, & b_2, & \dots, & b_q \end{smallmatrix} \right. \right)$ .

Let  $X(1), \dots, X(M)$  be independent matrices whose entries are i.i.d standard complex Gaussian variables. Assume that  $X(j)$  has size  $N_j \times N_{j-1}$ , where  $1 \leq j \leq n$ . Consider the product matrix

$$Y_M = X(M)X(M-1) \dots X(1),$$

which is a rectangular matrix of size  $N_M \times N_0$ . Note that  $Y_M^* Y_M$  is a square matrix of size  $N_0$ . Set

$$\nu_l = N_l - N_0, \quad l = 0, 1, \dots, M.$$

Akemann, Ipsen and Kieburg [6] proved the following

**Theorem 2.1.** *The squared singular values of  $Y_M$  form a determinantal point process on  $\mathbb{R}_{>0}$ . This determinantal point process is a biorthogonal ensemble<sup>1</sup> with joint density function given by*

$$P^{(M)}(x_1, \dots, x_{N_0}) = \frac{1}{Z_{N_0}} \prod_{1 \leq j < k \leq N_0} (x_k - x_j) \det \left( w_{k-1}^{(M)}(x_j) \right)_{j,k=1}^{N_0},$$

where  $x_k > 0$ ,  $k = 1, \dots, N_0$ , are the squared singular values of  $Y_M$ ,  $Z_{N_0}$  is a normalization constant, and the functions  $w_k^{(M)}(x)$  can be expressed in terms of Meijer's  $G$ -functions,

$$w_k^{(M)}(x) = G_{0,M}^{M,0} \left( x \left| \begin{smallmatrix} \nu_M, & \nu_{M-1}, & \dots, & \nu_2 & \nu_1 + k \end{smallmatrix} \right. \right).$$

<sup>1</sup>The notion of biorthogonal ensembles was introduced in Borodin [9]

As  $n = 1$  we obtain the determinantal point process on  $\mathbb{R}_{>0}$  called the classical Laguerre ensemble.

Kuijlaars and Zhang [22] found the scaling limit at the origin of the relevant correlation kernel which generalizes the classical Bessel kernel. According to Kuijlaars and Zhang [22], the limiting kernel can be written explicitly in terms of Meijer's G-functions. Namely, the following Theorem holds true

**Theorem 2.2.** *Let  $K_{N_0}^{(M)}(x, y)$  be the correlation kernel of the determinantal point process defined by the joint density  $P^{(M)}(x_1, \dots, x_{N_0})$  (see Theorem 2.1). Then we have*

$$\lim_{N_0 \rightarrow \infty} \frac{1}{N_0} K_{N_0}^{(M)} \left( \frac{x}{N_0}, \frac{y}{N_0} \right) = K_M(x, y),$$

where  $K_M(x, y)$  is given by

$$(2.1) \quad K_M(x, y) = \frac{\mathcal{B} \left( G_{0, M+1}^{1,0} \left( x \middle| -\nu_0, -\nu_1, \dots, -\nu_M \right), G_{0, M+1}^{M,0} \left( y \middle| \nu_1, \dots, \nu_M, \nu_0 \right) \right)}{x - y}.$$

In the formula above  $\mathcal{B}(\cdot, \cdot)$  is a bilinear operator defined by

$$\mathcal{B}(f(x), g(y)) = (-1)^{M+1} \sum_{j=0}^M (-1)^j \left( x \frac{d}{dx} \right)^j f(x) \left( \sum_{i=0}^{M-j} \alpha_{i+j} \left( y \frac{d}{dy} \right)^i g(y) \right).$$

The constants  $\alpha_i$  are determined by

$$(2.2) \quad \prod_{i=1}^M (x - \nu_i) = \sum_{i=0}^M \alpha_i x^i.$$

Using the fact that  $G_{0,2}^{1,0} \left( x \middle| \nu_1, \nu_2 \right)$  can be written in terms of the Bessel functions it is not hard to check (see Kuijlaars and Zhang [22], Section 5.3) that if  $M = 1$  and  $\nu_1 = \nu$ , one obtains a kernel equivalent to the classical Bessel kernel (equation (1.1)). It was observed in Kuijlaars and Zhang [22] that for  $M = 2$  kernel (2.1) coincides with the scaling limit found by Bertola, Gekhtman and Szmigielski [8] in the Cauchy-Laguerre two-matrix model. Moreover, Forrester [13] has proved that the kernel  $K_M(x, y)$  also arises (at the same scaling limit) in the problem on eigenvalue statistics for complex  $N \times N$  Wishart matrices  $W_{r,s}^* W_{r,s}$ , where  $W_{r,s}$  is the product of  $r$  complex Gaussian matrices, and the inverse of  $s$  complex Gaussian matrices.

Moreover, Kuijlaars and Zhang [22] noted that limiting kernel (2.1) can be represented in the form

$$(2.3) \quad K(x, y) = \frac{\sum_{i=1}^k F_i(x) G_i(y)}{x - y}, \quad \text{where} \quad \sum_{i=1}^k F_i(x) G_i(x) = 0.$$

Such kernels (and operators defined by such kernels) are called integrable, see Its, Izergin, Korepin and Slavnov [16], Deift [10]. Many questions in the theory of classical ensembles of random matrices can be reduced to the evaluation of Fredholm determinants  $\det(I - \lambda K)$ , where  $K$  is an operator which has a kernel of form (2.3). For a general theory of such Fredholm determinants see, for example, Its [17], Its and Harnad [18], Deift [10], Tracy and Widom [33, 34] and the references therein.

To summarize, the results obtained in Akemann, Ipsen and Kieburg [6], Kuijlaars and Zhang [22] imply that after rescaling at the hard edge (origin) the statistics of singular values is described by Fredholm determinants of integrable operators (in the sense of Its, Isergin, Korepin and Slavnov [16]).

### 3. MAIN RESULTS

**3.1. Notation.** Let  $f(x)$  and  $g(y)$  be two functions defined in terms of Meijer  $G$ -functions as follows

$$(3.1) \quad f(x) = G_{0,M+1}^{1,0} \left( x \middle| -\nu_0, -\nu_1, \dots, -\nu_M \right),$$

and

$$(3.2) \quad g(y) = G_{0,M+1}^{M,0} \left( y \middle| \nu_1, \dots, \nu_M, \nu_0 \right).$$

**Proposition 3.1.** *The functions  $f(x)$  and  $g(y)$  satisfy the following differential equations*

$$(3.3) \quad \prod_{j=0}^M (\Delta_x + \nu_j) f(x) = -x f(x),$$

and

$$(3.4) \quad \prod_{j=0}^M (\Delta_y - \nu_j) g(y) = (-1)^M y g(y),$$

where  $\Delta_x = x \frac{d}{dx}$ , and  $\Delta_y = y \frac{d}{dy}$ .

*Proof.* Use the fact that the Meijer  $G$ -function  $G_{p,q}^{m,n} \left( z \middle| \begin{smallmatrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{smallmatrix} \right)$  satisfies the following differential equation

$$(3.5) \quad \left[ (-1)^{p-m-n} z \prod_{j=1}^p \left( z \frac{d}{dz} - a_j + 1 \right) - \prod_{j=1}^q \left( z \frac{d}{dz} - b_j \right) \right] G_{p,q}^{m,n} \left( z \middle| \begin{smallmatrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{smallmatrix} \right) = 0,$$

see Ref. [28], formula (16.21.1). □

For  $0 \leq j \leq M$  define

$$(3.6) \quad \phi_j(x) = (-1)^{M-j+1} (\Delta_x)^j f(x), \quad \psi_j(y) = \sum_{i=0}^{M-j} \alpha_{i+j} (\Delta_y)^i g(y),$$

where the coefficients  $\alpha_0, \dots, \alpha_M$  are defined by equation (2.2). Using these functions, the correlation kernel  $K_M(x, y)$  (defined by equation (2.1)) can be written as

$$(3.7) \quad K_M(x, y) = \frac{\sum_{j=0}^M \phi_j(x) \psi_j(y)}{x - y},$$

where the indeterminacy arising for  $x = y$  is resolved via the L'Hospital rule. As it is noted in Kuijlaars and Zhang [22], Section 5.2, the kernel  $K_M(x, y)$  has the continuity property. This means that the condition

$$(3.8) \quad \sum_{j=0}^M \phi_j(x) \psi_j(x) = 0$$

holds true. In what follows we will refer to the kernel  $K_M(x, y)$  defined by equation (2.1) or, equivalently, by equation (3.7) as to the *generalized Bessel kernel*.

**Proposition 3.2.** *The correlation kernel  $K_M(x, y)$  can be written as*

$$(3.9) \quad K_M(x, y) = \int_0^1 f(xt)g(yt)dt.$$

*Proof.* See Kuijlaars and Zhang [22], Section 5.2. □

Let  $a_j \geq 0$ ,  $j = 1, \dots, 2m$ , and denote by  $J$  the union of intervals of the form  $(a_{2j-1}, a_{2j})$ . Thus  $J = \bigcup_{j=1}^m (a_{2j-1}, a_{2j})$ . Denote by  $K_M$  the operator acting in  $L^2(-\infty, +\infty)$  with the kernel  $K_M(x, y)\chi_J(y)$ , where  $\chi_J$  denotes the characteristic function of the interval  $J$ . In addition, denote by  $K'_M$  the operator with the kernel  $K_M(y, x)\chi_J(y)$ , and by  $K_M^T$  the operator with the kernel  $K_M(y, x)\chi_J(x)$ . Define the operators  $R_M$  and  $\rho_M$  by

$$(3.10) \quad R_M = (1 - K_M)^{-1}K_M = -1 + (1 - K_M)^{-1}, \quad \rho_M = (1 - K_M)^{-1}.$$

In the same way we define the operators  $R'_M$  and  $\rho'_M$ , i.e.

$$(3.11) \quad R'_M = (1 - K'_M)^{-1}K'_M = -1 + (1 - K'_M)^{-1}, \quad \rho'_M = (1 - K'_M)^{-1},$$

and the operators  $R_M^T$  and  $\rho_M^T$ ,

$$(3.12) \quad R_M^T = (1 - K_M^T)^{-1}K_M^T = -1 + (1 - K_M^T)^{-1}, \quad \rho_M^T = (1 - K_M^T)^{-1}.$$

**3.2. Partial differential equations.** For  $0 \leq j \leq M$  and  $1 \leq l \leq m$  define the following quantities

$$\begin{aligned} x_j^{(2l)} &:= \sqrt{-1} (I - K_M)^{-1} \phi_j(a_{2l}), \\ y_j^{(2l)} &:= \sqrt{-1} (I - K'_M)^{-1} \psi_j(a_{2l}), \\ x_j^{(2l-1)} &:= (I - K_M)^{-1} \phi_j(a_{2l-1}), \\ y_j^{(2l-1)} &:= (I - K'_M)^{-1} \psi_j(a_{2l-1}), \\ \xi_j &:= (-1)^M \sum_{k=1}^{2m} \int_{a_{2k-1}}^{a_{2k}} \phi_0(x) (I - K'_M)^{-1} \psi_j(x) dx + (-1)^{M+1-j} e_{M+1-j}(\nu_0, \dots, \nu_M), \\ \eta_j &:= (-1)^M \sum_{k=1}^{2m} \int_{a_{2k-1}}^{a_{2k}} \phi_j(x) (I - K'_M)^{-1} \psi_M(x) dx. \end{aligned}$$

These quantities are functions of parameters  $a_1, \dots, a_{2m}$ .

**Proposition 3.3.** *Functions  $x_j^{(k)}$ ,  $y_j^{(k)}$ ,  $\xi_j$  and  $\eta_j$  satisfy the following system of partial differential equations*

(a) *For  $0 \leq j \leq M$ , and  $1 \leq k \neq l \leq 2m$*

$$(3.13) \quad \frac{\partial x_j^{(l)}}{\partial a_k} = -\frac{x_j^{(k)}}{a_l - a_k} \sum_{i=0}^M x_i^{(l)} y_i^{(k)}.$$

(b) *For  $0 \leq j \leq M$ , and  $1 \leq k \neq l \leq 2m$*

$$(3.14) \quad \frac{\partial y_j^{(l)}}{\partial a_k} = -\frac{y_j^{(k)}}{a_k - a_l} \sum_{i=0}^M x_i^{(k)} y_i^{(l)}.$$

(c) *For  $0 \leq j \leq M-1$ , and  $1 \leq l \leq 2m$*

$$(3.15) \quad a_l \frac{\partial x_j^{(l)}}{\partial a_l} = -\eta_j x_0^{(l)} - x_{j+1}^{(l)} + \sum_{\substack{k=1 \\ k \neq l}}^{2m} x_j^{(k)} \frac{a_k}{a_l - a_k} \sum_{i=0}^M x_i^{(l)} y_i^{(k)},$$

and for  $j = M$  and  $l = 1, \dots, 2m$

$$(3.16) \quad \begin{aligned} a_l \frac{\partial x_M^{(l)}}{\partial a_l} &= -\eta_M x_0^{(l)} + (-1)^{M+1} a_l x_0^{(l)} \\ &+ \sum_{i=0}^M \xi_i x_i^{(l)} + \sum_{\substack{k=1 \\ k \neq l}}^{2m} x_M^{(k)} \frac{a_k}{a_l - a_k} \sum_{i=0}^M x_i^{(l)} y_i^{(k)}. \end{aligned}$$

(d) *For  $1 \leq j \leq M$ , and  $1 \leq l \leq 2m$*

$$(3.17) \quad a_l \frac{\partial y_j^{(l)}}{\partial a_l} = -\xi_j y_M^{(l)} + y_{j-1}^{(l)} + \sum_{\substack{k=1 \\ k \neq l}}^{2m} y_j^{(k)} \frac{a_k}{a_k - a_l} \sum_{i=0}^M x_i^{(k)} y_i^{(l)},$$

and for  $j = 0$  and  $l = 1, \dots, 2m$

$$(3.18) \quad \begin{aligned} a_l \frac{\partial y_0^{(l)}}{\partial a_l} &= -\xi_0 y_M^{(l)} + (-1)^M a_l y_M^{(l)} \\ &+ \sum_{i=0}^M \eta_i y_i^{(l)} + \sum_{\substack{k=1 \\ k \neq l}}^{2m} y_0^{(k)} \frac{a_k}{a_k - a_l} \sum_{i=0}^M x_i^{(k)} y_i^{(l)}. \end{aligned}$$

(e) *For  $0 \leq i, j \leq M$ , and  $1 \leq l \leq 2m$*

$$(3.19) \quad \frac{\partial}{\partial a_l} \xi_j = (-1)^{M+1} x_0^{(l)} y_j^{(l)}.$$

(f) *For  $0 \leq i, j \leq M$ , and  $1 \leq l \leq 2m$*

$$(3.20) \quad \frac{\partial}{\partial a_l} \eta_j = (-1)^{M+1} x_j^{(l)} y_M^{(l)}.$$

We will refer to equations (3.13)-(3.20) as to the system of dynamical equations associated with the generalized Bessel kernel  $K_M(x, y)$  defined by equation (3.7).

**Remarks.**

a) The quantity  $(1 - K_M)^{-1}\phi_j(a_k)$  means  $\lim_{\substack{x \in J \\ x \rightarrow a_k}} \phi_j(x)$ . The same meaning has the quantity  $(1 -$

$K'_M)\psi_j(a_k)$ .

b) For the sine kernel partial differential equations similar to that of Proposition 3.3 were first derived by Jimbo, Miwa, Mōri and Sato [19]. These equations are called the JMMS equations in the random matrix literature. Tracy and Widom [30]-[33] derived analogues of the JMMS equations for correlation kernels of the form

$$\frac{A(x)B(y) - A(y)B(x)}{x - y}.$$

Equations (3.13)-(3.20) of Proposition 3.3 are partial differential equations for a correlation kernel of even more general form (2.3).

c) Assume that  $M = 1$ . In this case

$$f(x) = g(x) = G_{0,2}^{1,0} \left( x \middle| \begin{matrix} 0, & 0 \end{matrix} \right) = J_0(2\sqrt{x}).$$

The functions  $\phi_0(x)$ ,  $\phi_1(x)$ ,  $\psi_0(y)$ , and  $\psi_1(y)$  are defined by equations

$$(3.21) \quad \phi_0(x) = f(x), \quad \phi_1(x) = -x \frac{d}{dx} f(x),$$

and

$$(3.22) \quad \psi_0(y) = y \frac{d}{dy} f(y), \quad \psi_1(y) = f(y).$$

In particular, we have

$$\phi_0 = \psi_1, \quad \phi_1 = -\psi_0,$$

and the kernel of  $K_{M=1}$  can be written as

$$K_{M=1}(x, y) \chi_J(y), \quad K_{M=1}(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y},$$

where the functions  $A(x)$ ,  $B(y)$  are defined by

$$A(x) = f(x), \quad B(y) = y \frac{d}{dy} f(y).$$

In this case the partial differential equations (3.13)-(3.20) turn into partial differential equations for the Bessel kernel derived in Tracy and Widom [32], see equations (1.9)-(1.14) of Tracy and Widom [32].

**3.3. Hamiltonian structure of dynamical equations associated with the generalized Bessel kernel.** Here we claim that the system of dynamical equations associated with the generalized Bessel kernel  $K_M(x, y)$  can be understood as a system of Hamiltonian equations.

For  $1 \leq l \leq 2m$  set

$$(3.23) \quad H_l := -a_l \frac{\partial}{\partial a_l} \log (\det (I - K_M)).$$



Introduce the Poisson brackets by

$$(3.24) \quad \left\{ x_j^{(l)}, y_i^{(k)} \right\} = \frac{1}{a_l} \delta_{l,k} \delta_{i,j}, \quad \left\{ \xi_j, \eta_i \right\} = \delta_{i,j}.$$

In other words, for two functions  $F$  and  $G$  of dynamical variables  $(x_j^{(k)}, \xi_j; y_j^{(k)}, \eta_j)$  (where  $1 \leq k \leq 2m$ , and  $0 \leq j \leq M$ ) we define the Poisson brackets by the formula

$$(3.25) \quad \{F, G\} := \sum_{k=1}^{2m} \frac{1}{a_k} \sum_{j=0}^M \left( \frac{\partial F}{\partial x_j^{(k)}} \frac{\partial G}{\partial y_j^{(k)}} - \frac{\partial F}{\partial y_j^{(k)}} \frac{\partial G}{\partial x_j^{(k)}} \right) + \sum_{j=0}^M \left( \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j} \frac{\partial G}{\partial \xi_j} \right).$$

**Proposition 3.4.** *The system of dynamical equations (3.13)-(3.20) associated with the generalized Bessel kernel  $K_M(x, y)$  (defined by equation (3.7)) can be written as*

$$(3.26) \quad \frac{\partial x_j^{(k)}}{\partial a_l} = \left\{ x_j^{(k)}, H_l \right\}, \quad \frac{\partial y_j^{(k)}}{\partial a_l} = \left\{ y_j^{(k)}, H_l \right\},$$

and

$$(3.27) \quad \frac{\partial \xi_j}{\partial a_l} = \left\{ \xi_j, H_l \right\}, \quad \frac{\partial \eta_j}{\partial a_l} = \left\{ \eta_j, H_l \right\},$$

where  $0 \leq j \leq M$ , and  $1 \leq k, l \leq 2m$ . The Hamiltonians are given explicitly by

$$(3.28) \quad \begin{aligned} H_l = & - \left( \sum_{j=0}^M \eta_j y_j^{(l)} \right) x_0^{(l)} + (-1)^{M+1} a_l x_0^{(l)} y_M^{(l)} \\ & - \sum_{j=0}^{M-1} x_{j+1}^{(l)} y_j^{(l)} + y_M^{(l)} \sum_{k=0}^M \xi_k x_k^{(l)} \\ & + \sum_{\substack{k=1 \\ k \neq l}}^{2m} \frac{a_k}{a_l - a_k} \sum_{i,j=0}^M x_i^{(k)} x_j^{(l)} y_i^{(l)} y_j^{(k)}. \end{aligned}$$

**Proposition 3.5.** *The Hamiltonians  $H_l$  are in involution. This means that for all  $1 \leq l, \rho \leq 2m$  the following condition is satisfied*

$$(3.29) \quad \{H_l, H_\rho\} = 0,$$

where the symplectic structure is defined by equation (3.24).

#### Remarks.

- a) A similar Hamiltonian interpretation is known for other systems of partial differential equations associated with correlation kernels of Random Matrix Theory, see Tracy and Widom [30]-[33].
- b) It is shown in Tracy and Widom [30] (in the context of the sine kernel) that the fact that the Hamiltonians are in involution implies the complete integrability of partial differential equations (3.26), (4.44) (or, equivalently, of equations (3.13)-(3.20)) in the sense of Frobenius. We will address this issue using isomonodromic interpretation of equations (3.26), (4.44), see Section 3.4.
- c) For each  $1 \leq l \leq m$  set

$$h_l = \frac{\partial}{\partial a_l} \log (\det (1 - K_M)).$$

Introduce the 1-form

$$w(a_1, \dots, a_{2m}) = \sum_{l=1}^{2m} h_l da_l.$$

Using the fact that the Hamiltonians  $H_l$  are in involution (see Proposition 3.5) we obtain that  $w(a_1, \dots, a_{2m})$  is locally an exact differential

$$w(a_1, \dots, a_{2m}) = d \log \tau.$$

This equation defines (up to a multiple constant) the tau-function associated with the dynamical equations (3.26) and (3.27). We conclude that this tau-function evaluated on a solution of the dynamical equations (3.26) and (3.27) is given by the Fredholm determinant

$$(3.30) \quad \tau(a_1, \dots, a_{2m}) = \det(1 - K_M).$$

**3.4. The isomonodromic system associated with the generalized Bessel kernel.** Set

$$(3.31) \quad E = (-1)^{M+1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

and

$$(3.32) \quad A^{(l)} = \begin{pmatrix} x_0^{(l)} y_0^{(l)} & x_0^{(l)} y_1^{(l)} & \dots & x_0^{(l)} y_M^{(l)} \\ x_1^{(l)} y_0^{(l)} & x_1^{(l)} y_1^{(l)} & \dots & x_1^{(l)} y_M^{(l)} \\ \vdots & & & \\ x_M^{(l)} y_0^{(l)} & x_M^{(l)} y_1^{(l)} & \dots & x_M^{(l)} y_M^{(l)} \end{pmatrix}, \quad C = \begin{pmatrix} -\eta_0 & -1 & 0 & \dots & 0 \\ -\eta_1 & 0 & -1 & \dots & 0 \\ \vdots & & & & \\ -\eta_{M-1} & 0 & 0 & \dots & -1 \\ \xi_0 - \eta_M & \xi_1 & \xi_2 & \dots & \xi_M \end{pmatrix},$$

where  $1 \leq l \leq 2m$ .

**Proposition 3.6.** *The system of dynamical equations (3.13)-(3.20) associated with the generalized Bessel kernel  $K_M(x, y)$  defined by equation (3.7) is equivalent to*

$$(3.33) \quad \frac{\partial}{\partial a_k} A^{(l)} = \frac{[A^{(l)}, A^{(k)}]}{a_l - a_k}, \quad 1 \leq l \neq k \leq 2m,$$

$$(3.34) \quad a_l \frac{\partial A^{(l)}}{\partial a_l} = [C + a_l E, A^{(l)}] + \sum_{\substack{k=1 \\ k \neq l}}^{2m} \frac{a_k}{a_l - a_k} [A^{(k)}, A^{(l)}], \quad 1 \leq l \leq 2m,$$

$$(3.35) \quad \frac{\partial C}{\partial a_l} = [E, A^{(l)}], \quad 1 \leq l \leq 2m.$$

Moreover, we have

$$H_l = \text{Tr} \left( C A^{(l)} \right) + a_l \text{Tr} \left( E A^{(l)} \right) + \sum_{\substack{k=1 \\ k \neq l}}^{2m} \frac{a_k}{a_l - a_k} \text{Tr} \left( A^{(k)} A^{(l)} \right),$$

where  $1 \leq l \leq 2m$ .

Consider the following linear system of ordinary differential equations with rational coefficients

$$(3.36) \quad \frac{d\Psi}{dz} = X(z)\Psi,$$

where  $X(z)$  is a  $(M+1) \times (M+1)$  matrix defined by

$$(3.37) \quad X(z) = E - \frac{C - \sum_{k=1}^{2m} A^{(k)}}{z} + \sum_{k=1}^{2m} \frac{A^{(k)}}{z - a_k},$$

and where  $E, C, A^{(k)}$  are certain  $(M+1) \times (M+1)$  matrices which are independent on  $z$ . Consider the poles  $a_1, \dots, a_{2m}$  as deformation parameters of the coefficient matrix  $X(z)$ . Thus

$$(3.38) \quad X(z) = X(z; a_1, \dots, a_{2m}).$$

Suppose that  $\Psi(z; a_1, \dots, a_{2m})$  (in addition to equation (3.36)) also satisfies a linear system of ordinary differential equations with respect to parameters  $a_1, \dots, a_{2m}$

$$(3.39) \quad \frac{\partial \Psi}{\partial a_j} = \Theta_j(z)\Psi, \quad \Theta_j(z) = - \sum_{k=1}^{2m} \frac{A^{(k)}}{z - a_k}, \quad 1 \leq j \leq 2m.$$

Note that we can write equations (3.39) as

$$(3.40) \quad d\Psi = \Theta(z)\Psi, \quad \Theta(z) = \sum_{j=1}^{2m} \Theta_j(z) da_j.$$

The compatibility condition of equations (3.36) and (3.40) is

$$(3.41) \quad dX = \frac{\partial \Theta}{\partial z} + [\Theta, X].$$

This equation is called the *isomonodromy deformation equation* in the Jimbo, Miwa, Mōri, Ueno and Sato theory [19]-[21] of isomonodromy deformations, and it gives a condition for deformation (3.38) to be isomonodromic. For the modern presentation of the theory of isomonodromy deformations we refer the reader to the book by Fokas, Its, Kapaev and Novokshenov [11], Chapter 4. Equating the corresponding principal parts in the isomonodromy deformation equation (3.41), we obtain equations (3.33)-(3.35) for the matrix coefficients  $A^{(k)}$ ,  $k = 1, \dots, 2m$ , and  $C$ . This is a crucial observation which implies that the following statement holds true.

**Proposition 3.7.** *The dynamical partial differential equations associated with the generalized Bessel kernel (equations (3.13)-(3.20), or, equivalently, equations (3.26)-(3.27), or, equivalently, equations (3.33)-(3.35)) can be written as the isomonodromy deformation equation (3.41), with matrices  $X(z)$  and  $\Theta(z)$  defined by equations (3.37) and (3.39).*

Now, let us recall the notion of the (Frobenius) complete integrability. Consider the following system of first-order partial differential equations

$$(3.42) \quad \frac{\partial y_j(b_1, \dots, b_n)}{\partial b_l} = \varphi_{j,l}(y_1(b_1, \dots, b_n), \dots, y_N(b_1, \dots, b_n), b_1, \dots, b_n),$$

where  $1 \leq j \leq N$ , and  $1 \leq l \leq n$ . Here  $(b_1, \dots, b_n) \in \mathbb{C}^n$  are independent variables,  $y_1, \dots, y_N$  are unknown functions of  $(b_1, \dots, b_n)$ , and  $\varphi_{j,l}$  are given holomorphic functions defined in a domain  $\mathcal{D} \subset \mathbb{C}^N \times \mathbb{C}^n$ .

**Definition 3.8.** The system of first-order partial differential equations (3.42) is called completely integrable (in the sense of Frobenius) if for any  $(z_1, \dots, z_N, \zeta_1, \dots, \zeta_n) \in \mathcal{D}$  there exists a solution of (3.42) such that

$$y_j(\zeta_1, \dots, \zeta_n) = z_j, \quad 1 \leq j \leq N.$$

It is known that isomonodromy deformation equations associated with systems of linear ordinary differential equations with rational coefficients are completely integrable in the sense of Frobenius. Thus we conclude that the dynamical partial differential equations associated with the generalized Bessel kernel (equations (3.13)-(3.20), or, equivalently, equations (3.26)-(3.27), or, equivalently, equations (3.33)-(3.35)) are completely integrable in the sense of Frobenius, see Jimbo, Miwa, and Ueno [20].

**Remark.**

The system

$$(3.43) \quad \begin{cases} \frac{d\Psi}{dz} = X(z)\Psi \\ d\Psi = \Theta(z)\Psi \end{cases}$$

(where the matrices  $X(z)$  and  $\Theta(z)$  defined by equations (3.37) and (3.39), and the matrices  $E$ ,  $C$ ,  $A^{(l)}$  are defined by equation (3.31) and equation (3.32)) can be understood as a Lax representation of the dynamical partial differential equations associated with the generalized Bessel kernel  $K_M(x, y)$ .

**3.5. The special case of  $J = (0, s)$ .** This case corresponds to  $m = 1$ ,  $a_1 = 0$ ,  $a_2 = s$ .

**Proposition 3.9. (A)** *In the special case  $J = (0, s)$  partial differential equations (3.13)-(3.20) lead to the following system of non-linear ordinary differential equations*

$$(3.44) \quad s \frac{dx_j(s)}{ds} = -\eta_j(s)x_0(s) - x_{j+1}(s), \quad 0 \leq j \leq M-1,$$

$$(3.45) \quad s \frac{dx_M(s)}{ds} = -\eta_M(s)x_0(s) + (-1)^{M+1}sx_0(s) + \sum_{i=0}^M \xi_i(s)x_i(s),$$

$$(3.46) \quad s \frac{dy_j(s)}{ds} = -\xi_j(s)y_M(s) + y_{j-1}(s), \quad 1 \leq j \leq M,$$

$$(3.47) \quad s \frac{dy_0(s)}{ds} = -\xi_0(s)y_M(s) + (-1)^M sy_M(s) + \sum_{i=0}^M \eta_i(s)y_i(s),$$

$$(3.48) \quad \frac{d\xi_i(s)}{ds} = (-1)^{M+1}x_0(s)y_j(s), \quad 0 \leq j \leq M,$$

$$(3.49) \quad \frac{d\eta_j(s)}{ds} = (-1)^{M+1}x_j(s)y_M(s), \quad 0 \leq j \leq M.$$

**(B)** *Set*

$$F_M(s) = \det(1 - K_M),$$

where  $K_M$  is the operator with the kernel  $K_M(x, y)\chi_{(0, s)}(y)$  acting on  $L^2((-\infty, +\infty))$ . We have

$$F_M(s) = \exp \left( (-1)^M \int_0^s \log \left( \frac{s}{t} \right) x_0(t) y_M(t) dt \right),$$

where  $x_0(t)$ ,  $y_M(t)$  are components of the solution  $(x_j(t), y_j(t), \xi_j(t), \eta_j(t))$  of non-linear ordinary differential equations (3.44)-(3.49) with the initial conditions

$$x_j(0) = y_j(0) = \eta_j(0) = 0, \quad \xi_j(0) = (-1)^{M+1-j} e_{M+1-j}(\nu_0, \dots, \nu_M), \quad 0 \leq j \leq M.$$

**Remarks.**

a) Note that in the case of a single interval  $(0, s)$  equations (3.33)- (3.35) take an especially simple form. Namely, we have

$$(3.50) \quad s \frac{d}{ds} A = [C + sE, A], \quad \frac{d}{ds} C = [E, A].$$

The Hamiltonian is given by

$$(3.51) \quad H = \text{Tr}(CA) + s \text{Tr}(EA).$$

b) The Fredholm determinant  $F_M(s)$  defined above gives the probability that no particles of the determinantal point process defined by the generalized Bessel kernel (3.7) lie in the interval  $(0, s)$ .  
 c) In the case  $M = 1$  the functions  $x_0(t)$  and  $y_M(t)$  coincide with each other, and we obtain the same result as in Tracy and Widom [32] (equation (1.19)). In this case equations (3.44)-(3.49) lead to a single ordinary differential equation for  $x_0(t)$  (or for  $y_M(t)$ ) which is reducible to a special case of fifth Painlevé equation.

#### 4. PROOFS

Our first task is to prove Proposition 3.3. This can be done by adopting methods developed in Tracy and Widom [30]-[34] to the situation where the operators are defined by the generalized Bessel kernel  $K_M(x, y)$  (equation(3.7)). Note that the functions  $\phi_j, \psi_j$  in formula (3.7) are given explicitly in terms of Meijer's  $G$ -functions, see equations (3.1), (3.2), and (3.6).

**4.1. The functions  $\mathcal{Q}_j(x; a_1, \dots, a_{2m})$ ,  $\mathcal{P}_j(x; a_1, \dots, a_{2m})$ , and  $V_{i,j}(a_1, \dots, a_{2m})$ .** For  $0 \leq j \leq M$  introduce the following functions

$$(4.1) \quad \mathcal{Q}_j(x; a_1, \dots, a_{2m}) = (1 - K_M)^{-1} \phi_j(x),$$

and

$$(4.2) \quad \mathcal{P}_j(x; a_1, \dots, a_{2m}) = (1 - K'_M)^{-1} \psi_j(x).$$

In addition, for  $0 \leq i, j \leq M$  set

$$(4.3) \quad V_{i,j}(a_1, \dots, a_{2m}) = \int_J \phi_i(x) \mathcal{P}_j(x; a_1, \dots, a_{2m}) dx.$$

**Proposition 4.1.** *The functions  $\mathcal{Q}_j(x; a_1, \dots, a_{2m})$  defined by equation (4.1) satisfy a system of partial differential equations. Namely, for  $0 \leq j \leq M - 1$  we have*

$$(4.4) \quad \begin{aligned} x \frac{\partial}{\partial x} \mathcal{Q}_j(x; a_1, \dots, a_{2m}) &= (-1)^{M+1} V_{j,M}(a_1, \dots, a_{2m}) \mathcal{Q}_0(x; a_1, \dots, a_{2m}) \\ &- \mathcal{Q}_{j+1}(x; a_1, \dots, a_{2m}) - \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \mathcal{Q}_j(a_k; a_1, \dots, a_{2m}), \end{aligned}$$

and for  $j = M$  we have

$$\begin{aligned}
 x \frac{\partial}{\partial x} \mathcal{Q}_M(x; a_1, \dots, a_{2m}) &= (-1)^{M+1} V_{M,M}(a_1, \dots, a_{2m}) \mathcal{Q}_0(x; a_1, \dots, a_{2m}) \\
 &+ (-1)^{M+1} x \mathcal{Q}_0(x; a_1, \dots, a_{2m}) + \sum_{k=0}^M (-1)^{M+1-k} e_{M+1-k}(\nu_0, \dots, \nu_M) \mathcal{Q}_k(x; a_1, \dots, a_{2m}) \\
 (4.5) \quad &+ (-1)^M \sum_{k=0}^M V_{0,k}(a_1, \dots, a_{2m}) \mathcal{Q}_k(x; a_1, \dots, a_{2m}) \\
 &- \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \mathcal{Q}_M(a_k; a_1, \dots, a_{2m}).
 \end{aligned}$$

In addition, for  $1 \leq k \leq 2m$ , and for  $0 \leq j \leq M$  we have

$$(4.6) \quad \frac{\partial}{\partial a_k} \mathcal{Q}_j(x; a_1, \dots, a_{2m}) = (-1)^k R_M(x, a_k) \mathcal{Q}_j(a_k; a_1, \dots, a_{2m}).$$

In the formulae just written above  $R_M(x, a_k)$  stands for the kernel of  $R_M$  at points  $x$  and  $a_k$ .

*Proof.* Let us denote by  $D$  the operator of differentiation, and by  $M$  the operator of multiplication. Thus we have

$$D\varphi(x) = \frac{d}{dx}\varphi(x), \quad M\varphi(x) = x\varphi(x).$$

With this notation we have

$$\begin{aligned}
 (4.7) \quad MD\mathcal{Q}_j(x) &= MD(1 - K_M)^{-1}\phi_j(x) \\
 &= [MD, (1 - K_M)^{-1}]\phi_j(x) + (1 - K_M)^{-1}MD\phi_j(x), \quad 0 \leq j \leq M.
 \end{aligned}$$

Let us first compute the first term in the right-hand side of the equation just written above. We use the identity

$$(4.8) \quad [MD, (1 - K_M)^{-1}] = (1 - K_M)^{-1}[MD, K_M](1 - K_M)^{-1}.$$

Thus we need a formula for the commutator  $[MD, K_M]$ . To find such a formula observe that for any operator  $L$  with the kernel  $L(x, y)$  the following identity holds true

$$(4.9) \quad [MD, L](x, y) = ((MD)_x + (MD)_y + I)L(x, y).$$

Using this identity we obtain

$$\begin{aligned}
 (4.10) \quad [MD, K_M](x, y) &= \left( \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) K_M(x, y) \right) \chi_J(y) \\
 &+ y \left( \frac{\partial}{\partial y} \chi_J(y) \right) K_M(x, y) + K_M(x, y) \chi_J(y).
 \end{aligned}$$

Using formula (3.9) we find that

$$\begin{aligned}
 (4.11) \quad \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) K_M(x, y) &= \int_0^1 t \frac{\partial}{\partial t} (f(tx)g(ty)) dt \\
 &= f(x)g(y) - K_M(x, y) = (-1)^{M+1} \phi_0(x) \psi_M(y) - K_M(x, y).
 \end{aligned}$$

In addition, we have

$$(4.12) \quad \frac{\partial}{\partial y} \chi_J(y) = \sum_{k=1}^{2m} (-1)^{k-1} \delta(y - a_k).$$

Equations (4.10), (4.11), and (4.12) give us

$$(4.13) \quad [MD, K_M](x, y) = (-1)^{M+1} \phi_0(x) \psi_M(y) \chi_J(y) - \sum_{k=1}^{2m} (-1)^k a_k K_M(x, a_k) \delta(y - a_k).$$

Using identity (4.8) together with formula (4.13) we find the kernel of  $[MD, (1 - K_M)^{-1}]$ . Namely,

$$(4.14) \quad [MD, (1 - K_M)^{-1}](x, y) = (-1)^{M+1} Q_0(x) \mathcal{P}_M^*(y) - \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \rho_M(a_k, y),$$

where  $\mathcal{P}_M^*(y)$  is defined by

$$\mathcal{P}_M^*(y) = (1 - K_M^T)^{-1} \widetilde{\psi_M}(y), \quad \widetilde{\psi_M}(y) = \psi_M(y) \chi_J(y).$$

Therefore,

$$(4.15) \quad [MD, (1 - K_M)^{-1}] \phi_j(x) = (-1)^{M+1} Q_0(x) (\mathcal{P}_M^*, \phi_j) - \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \mathcal{Q}_j(a_k).$$

Note that

$$(4.16) \quad \mathcal{P}_M^*(y) = \mathcal{P}_M(y) \chi_J(y),$$

as it can be seen from the very definition of the operators  $K_M^T$  and  $K_M'$ . Thus we arrive to the formula

$$(4.17) \quad \begin{aligned} [MD, (1 - K_M)^{-1}] \phi_j(x) &= (-1)^{M+1} Q_0(x) V_{j,M}(a_1, \dots, a_{2m}) \\ &\quad - \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \mathcal{Q}_j(a_k), \end{aligned}$$

which holds true for all  $0 \leq j \leq M$ .

Now let us compute  $(1 - K_M)^{-1} MD \phi_j(x)$ . Equation (3.6) implies

$$(4.18) \quad (1 - K_M)^{-1} MD \phi_j(x) = -\mathcal{Q}_{j+1}(x), \quad 0 \leq j \leq M-1.$$

Thus equation (4.4) in the statement of the Proposition follows from equations (4.7), (4.17) and (4.18).

Let us consider the case when  $j = M$ . The definition of functions  $\phi_j(x)$  (see equation (3.6)) implies

$$MD \phi_M(x) = -(\Delta_x)^{M+1} f(x).$$

Equation (3.3) can be written as

$$(\Delta_x)^{M+1} f(x) = (-1)^M x \phi_0(x) + \sum_{k=0}^M (-1)^{M-k} e_{M+1-k}(\nu_0, \dots, \nu_M) \phi_k(x),$$

so

$$MD \phi_M(x) = (-1)^{M+1} x \phi_0(x) + \sum_{k=0}^M (-1)^{M-k+1} e_{M+1-k}(\nu_0, \dots, \nu_M) \phi_k(x),$$

and

$$(4.19) \quad \begin{aligned} (1 - K_M)^{-1} M D \phi_M(x) &= (-1)^{M+1} (1 - K_M)^{-1} M \phi_0(x) \\ &+ \sum_{k=0}^M (-1)^{M-k+1} e_{M+1-k}(\nu_0, \dots, \nu_M) \mathcal{Q}_k(x). \end{aligned}$$

It remains to compute  $(1 - K_M)^{-1} M \phi_0(x)$ . We have

$$(4.20) \quad \begin{aligned} (1 - K_M)^{-1} M \phi_0(x) &= [(1 - K_M)^{-1}, M] \phi_0(x) + M(1 - K_M)^{-1} \phi_0(x) \\ &= [(1 - K_M)^{-1}, M] \phi_0(x) + x \mathcal{Q}_0(x). \end{aligned}$$

Moreover, using the formula

$$[(1 - K_M)^{-1}, M] = (1 - K_M)^{-1} [K_M, M] (1 - K_M)^{-1},$$

and the fact that

$$[K_M, M](x, y) = - \sum_{j=0}^M \phi_j(x) \psi_J(y) \chi_J(y),$$

we obtain

$$[(1 - K_M)^{-1}, M](x, y) = - \sum_{j=0}^M \mathcal{Q}_j(x) \mathcal{P}_j(y) \chi_J(y).$$

So,

$$(4.21) \quad [(1 - K_M)^{-1} M] \phi_0(x) = - \sum_{j=0}^M \mathcal{Q}_j(x) V_{0,j}(a_1, \dots, a_{2m}).$$

Inserting (4.20) and (4.21) into equation (4.19) gives

$$(4.22) \quad \begin{aligned} (1 - K_M)^{-1} M D \phi_M(x) &= (-1)^{M+1} x \mathcal{Q}_0(x) + (-1)^M \sum_{j=0}^M \mathcal{Q}_j(x) V_{0,j}(a_1, \dots, a_{2m}) \\ &+ \sum_{k=0}^M (-1)^{M-k+1} e_{M+1-k}(\nu_0, \dots, \nu_M) \mathcal{Q}_k(x). \end{aligned}$$

Equation (4.5) in the statement of the Proposition follows from equations (4.7), (4.17), and (4.22).

It remains to derive equation (4.6). We have

$$(4.23) \quad \frac{\partial}{\partial a_k} \mathcal{Q}_j(x; a_1, \dots, a_{2m}) = \frac{\partial}{\partial a_k} (1 - K_M)^{-1} \phi_j(x).$$

Now,

$$\frac{\partial}{\partial a_k} (1 - K_M)^{-1} = (1 - K_M)^{-1} \frac{\partial K_M}{\partial a_k} (1 - K_M)^{-1},$$

and

$$\frac{\partial K_M}{\partial a_k}(x, y) = (-1)^k K_M(x, a_k) \delta(y - a_k).$$

Thus the kernel of  $\frac{\partial}{\partial a_k} (1 - K_M)^{-1}$  can be written as

$$(4.24) \quad \frac{\partial}{\partial a_k} (1 - K_M)^{-1}(x, y) = (-1)^k R_M(x, a_k) \rho_M(a_k, y),$$



and we arrive to equation (4.6). □

**Proposition 4.2.** *The partial differential equations for the functions  $\mathcal{P}_j(y; a_1, \dots, a_{2m})$  defined by equation (4.2) can be written as follows. For  $j = 0$  we have*

$$\begin{aligned}
 (4.25) \quad & y \frac{\partial}{\partial y} \mathcal{P}_0(y; a_1, \dots, a_{2m}) = (-1)^{M+1} V_{0,0}(a_1, \dots, a_{2m}) \mathcal{P}_m(y; a_1, \dots, a_{2m}) \\
 & + (-1)^M y \mathcal{P}_M(y; a_1, \dots, a_{2m}) + (-1)^M \sum_{j=0}^M V_{j,M}(a_1, \dots, a_{2m}) \mathcal{P}_j(y; a_1, \dots, a_{2m}) \\
 & + \sum_{k=1}^{2m} (-1)^{k-1} a_k R'_M(y, a_k) \mathcal{P}_0(a_k; a_1, \dots, a_{2m}),
 \end{aligned}$$

and for  $1 \leq j \leq M$  we have

$$\begin{aligned}
 (4.26) \quad & y \frac{\partial}{\partial y} \mathcal{P}_j(y; a_1, \dots, a_{2m}) = \mathcal{P}_{j-1}(y; a_1, \dots, a_{2m}) \\
 & + (-1)^{M-j} e_{M-j+1}(\nu_1, \dots, \nu_M) \mathcal{P}_M(y; a_1, \dots, a_{2m}) \\
 & + (-1)^{M+1} V_{0,j}(a_1, \dots, a_{2m}) \mathcal{P}_M(y; a_1, \dots, a_{2m}) \\
 & + \sum_{k=1}^{2m} (-1)^{k-1} a_k R'_M(y, a_k) \mathcal{P}_j(a_k; a_1, \dots, a_{2m}).
 \end{aligned}$$

In addition, for all  $k$ ,  $1 \leq k \leq 2m$ , we have

$$(4.27) \quad \frac{\partial}{\partial a_k} \mathcal{P}_j(y; a_1, \dots, a_{2m}) = (-1)^k R'_M(y, a_k) \mathcal{P}_j(a_k; a_1, \dots, a_{2m}).$$

*Proof.* We have

$$(4.28) \quad MD\mathcal{P}_j(x) = [MD, (1 - K'_M)^{-1}] \psi_j(x) + (1 - K'_M)^{-1} MD\psi_j(x), \quad 0 \leq j \leq M.$$

Calculations similar to that leading to formula (4.17) give us

$$\begin{aligned}
 (4.29) \quad & [MD, (1 - K'_M)^{-1}] \psi_j(x) = (-1)^{M+1} \mathcal{P}_M(x) V_{0,j}(a_1, \dots, a_{2m}) \\
 & + \sum_{k=1}^{2m} (-1)^{k-1} a_k R'_M(x, a_k) \mathcal{P}_j(a_k), \quad 0 \leq j \leq M.
 \end{aligned}$$

Note that equations (3.4) and (3.6) imply

$$(4.30) \quad MD\psi_j(y) = \begin{cases} \psi_{j-1}(y) - \alpha_{j-1} \psi_M(y), & 1 \leq j \leq M, \\ (-1)^M y \psi_M(y), & j = 0. \end{cases}$$

It follows immediately from equation (4.30) that

$$(4.31) \quad (1 - K'_M)^{-1} MD\psi_j(y) = \mathcal{P}_{j-1}(y) + (-1)^{M-j} e_{M-j+1}(\nu_1, \dots, \nu_M) \mathcal{P}_M(y),$$

where  $1 \leq j \leq M$ . Equations (4.29) and (4.31) give equation (4.27) in the statement of the Proposition. Moreover,

$$\begin{aligned}
 (4.32) \quad & (1 - K'_M)^{-1} MD\psi_0(y) = (-1)^M (1 - K'_M)^{-1} \psi_M(y) \\
 & = (-1)^M y \mathcal{P}_M(y) + (-1)^M [(1 - K'_M)^{-1}, M] \psi_M(y).
 \end{aligned}$$

After straightforward calculations (similar to those in the proof of Proposition 4.1) we obtain

$$[K'_M, M](x, y) = \sum_{j=0}^M \psi_j(x) \phi_j(y) \chi_J(y),$$

and

$$(4.33) \quad \begin{aligned} [(1 - K'_M)^{-1}, M] \psi_M(y) &= (1 - K'_M)^{-1} [K'_M, M] (1 - K'_M)^{-1} \psi_M(y) \\ &= \sum_{j=0}^M \mathcal{P}_j(y) V_{j,M}(a_1, \dots, a_{2m}). \end{aligned}$$

Equation (4.25) follows from equations (4.28), (4.29), (4.32), and (4.33). In order to see that equation (4.27) holds true use the formula

$$\frac{\partial}{\partial a_k} (1 - K'_M)^{-1} = (1 - K'_M)^{-1} \frac{\partial K'_M}{\partial a_k} (1 - K'_M)^{-1}.$$

The kernel of  $\frac{\partial}{\partial a_k} K'_M$  is

$$(-1)^k K_M(a_k, x) \delta(y - a_k),$$

and equation (4.27) follows. □

**Proposition 4.3.** *We have*

$$(4.34) \quad \frac{\partial}{\partial a_l} V_{i,j}(a_1, \dots, a_{2m}) = (-1)^l \mathcal{Q}_i(a_l; a_1, \dots, a_{2m}) \mathcal{P}_j(a_l; a_1, \dots, a_{2m}),$$

for  $0 \leq i, j \leq M$ , and for  $1 \leq l \leq 2m$ .

*Proof.* We note that

$$\frac{\partial}{\partial a_l} \chi_J(x) = (-1)^l \delta(x - a_l).$$

Taking into account this equation, together with equation (4.27), we obtain

$$(4.35) \quad \begin{aligned} \frac{\partial}{\partial a_l} V_{i,j}(a_1, \dots, a_{2m}) &= (-1)^l \phi_i(a_l) \mathcal{P}_j(a_l; a_1, \dots, a_{2m}) \\ &+ (-1)^l \left( \int_J \phi_i(x) R'_M(x, a_l) dx \right) \mathcal{P}_j(a_l; a_1, \dots, a_{2m}). \end{aligned}$$

Formula (4.34) follows from equations (4.35), (3.11), and from the fact that the kernel of  $K'_M$  (at points  $x, y$ ) (4.27) is  $K_M(y, x) \chi_J(y)$ . □

#### 4.2. Explicit formulae for the kernels $R_M(x, y)$ and $R'_M(x, y)$ .

**Proposition 4.4.** *Let  $R_M$  be the resolvent of  $K_M$ , and  $R'_M$  be the resolvent of  $K'_M$ . Denote by  $R_M(x, y)$  the kernel of  $R_M$ , and denote by  $R'_M(x, y)$  the kernel of  $R'_M$ . We have*

$$R_M(x, y) = \frac{\sum_{j=0}^M \mathcal{Q}_j(x; a_1, \dots, a_{2m}) \mathcal{P}_j(y; a_1, \dots, a_{2m})}{x - y} \chi_J(y),$$

and

$$R'_M(x, y) = -\frac{\sum_{j=0}^M \mathcal{P}_j(x; a_1, \dots, a_{2m}) \mathcal{Q}_j(y; a_1, \dots, a_{2m})}{x - y} \chi_J(y).$$

*Proof.* Recall that the operators  $R_M$  and  $R'_M$  are defined by equations (3.10), (3.11) correspondingly, where  $K_M$  is the operator with the kernel

$$(4.36) \quad K_M(x, y) \chi_J(y) = \frac{\sum_{j=0}^M \phi_j(x) \psi_j(y)}{x - y} \chi_J(y),$$

and where  $K'_M$  is the operator with the kernel

$$(4.37) \quad K_M(y, x) \chi_J(y) = -\frac{\sum_{j=0}^M \psi_j(x) \phi_j(y)}{x - y} \chi_J(y).$$

The kernels just written above are integrable in the sense of Its, Isergin, Korepin and Slavnov [16]. This implies that the corresponding resolvent kernel has the same integrable form. More explicitly,

$$(4.38) \quad R_M(x, y) = \frac{\sum_{j=0}^M ((1 - K_M)^{-1} \phi_j)(x) ((1 - K_M^T)^{-1} \psi_j \chi_J)(y)}{x - y},$$

and

$$(4.39) \quad R'_M(x, y) = -\frac{\sum_{j=0}^M ((1 - K'_M)^{-1} \psi_j)(x) ((1 - (K'_M)^T)^{-1} \phi_j \chi_J)(y)}{x - y}.$$

Using equations (4.36) and (4.37) we find

$$(4.40) \quad ((1 - K_M^T)^{-1} \psi_j \chi_J)(y) = \mathcal{P}_j(y; a_1, \dots, a_{2m}) \chi_J(y),$$

and

$$(4.41) \quad ((1 - (K'_M)^T)^{-1} \phi_j \chi_J)(y) = \mathcal{Q}_j(y; a_1, \dots, a_{2m}) \chi_J(y).$$

Now, equations (4.38), (4.1) and (4.40) give the formula for  $R_M(x, y)$  in the statement of the Proposition. In a similar way, we obtain the formula for  $R'_M(x, y)$  from equations (4.39), (4.2), and (4.41).  $\square$

**Proposition 4.5.** *Define the 1-form  $w(a_1, \dots, a_{2m})$  by the formula*

$$w(a_1, \dots, a_{2m}) = \sum_{j=1}^{2m} \frac{\partial}{\partial a_j} (\log \det (1 - K_M)) da_j.$$

*We have*

$$w(a_1, \dots, a_{2m}) = \sum_{j=1}^{2m} (-1)^{j-1} R_M(a_j, a_j) da_j.$$

*Proof.* Use the well-known formula

$$\frac{\partial}{\partial a_j} (\log \det (1 - K_M)) = -\operatorname{Tr} \left( (1 - K_M)^{-1} \frac{\partial}{\partial a_j} K_M \right),$$

and observe that the kernel of  $(1 - K_M)^{-1} \frac{\partial}{\partial a_j} K_M$  is

$$(-1)^j R_M(x, a_j) \delta(y - a_j).$$

□

**Proposition 4.6.** *The kernel of  $R_M$ ,  $R_M(x, y)$ , satisfies the following partial differential equation*

$$(4.42) \quad \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \sum_{k=1}^{2m} a_k \frac{\partial}{\partial a_k} + I \right) R_M(x, y) \\ = (-1)^{M+1} \mathcal{Q}_0(x; a_1, \dots, a_{2m}) \mathcal{P}_M(y; a_1, \dots, a_{2m}) \chi_J(y).$$

*Proof.* Formulae (4.9), (4.14), and (4.16) give us

$$\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + I \right) R_M(x, y) \\ = (-1)^{M+1} \mathcal{Q}_0(x; a_1, \dots, a_{2m}) \mathcal{P}_M(y; a_1, \dots, a_{2m}) \chi_J(y) - \sum_{k=1}^{2m} (-1)^k a_k R_M(x, a_k) \rho_M(a_k, y).$$

Moreover,

$$\sum_{k=1}^{2m} a_k \frac{\partial}{\partial a_k} R_M(x, y) = \sum_{k=1}^{2m} (-1)^K R_M(x, a_k) \rho_M(a_k, y),$$

as it follows from equation (4.24). □

**4.3. Proof of Proposition 3.3.** In order to obtain partial differential equations (3.13)-(3.20) in Proposition 3.3 we note that

$$x_j^{(2l)} = \sqrt{-1} \mathcal{Q}_j(a_{2l}; a_1, \dots, a_{2m}),$$

$$y_j^{(2l)} = \sqrt{-1} \mathcal{P}_j(a_{2l}; a_1, \dots, a_{2m}),$$

$$x_j^{(2l-1)} = \mathcal{Q}_j(a_{2l-1}; a_1, \dots, a_{2m}),$$

$$y_j^{(2l-1)} = \mathcal{P}_j(a_{2l-1}; a_1, \dots, a_{2m}),$$

and that

$$\xi_j = (-1)^M V_{0,j}(a_1, \dots, a_{2m}) + (-1)^{M+1-j} e_{M+1-j}(\nu_0, \dots, \nu_M),$$

$$\eta_j = (-1)^M V_{j,M}(a_1, \dots, a_{2m}).$$

In these formulae  $0 \leq j \leq M$ , and  $1 \leq l \leq m$ . Now use the results of Propositions 4.1, 4.2, 4.3, together with the explicit formulae for the kernels  $R_M(x, y)$  and  $R'_M(x, y)$  obtained in Proposition (4.4). Proposition 3.3 is proved. □

**4.4. Proof of Proposition 3.4, Proposition 3.5 and of Proposition 3.6.** To check equations (3.26) and (3.27) we need an explicit formula for the Hamiltonians  $H_l$ . From the very definition of  $H_l$  (equation (3.23)), and from Proposition 4.5 it follows that

$$H_l = (-1)^l a_l R_M(a_l, a_l), \quad 1 \leq l \leq 2m.$$

The quantity  $R_M(a_l, a_l)$  can be obtained using formulas derived in Propositions 4.1, 4.2, and in Proposition 4.4. The result is given by formula (3.28).

Using equation (3.28) we can check that equations (3.26) and (3.27) are in fact equivalent to the partial differential equations (3.13)-(3.20) in Proposition 3.3. This proves Proposition 3.4. In order to prove Proposition 3.5 note that equations (3.26) and (3.27) can be rewritten as

$$(4.43) \quad a_k \frac{\partial x_j^{(k)}}{\partial a_l} = \frac{\partial H_l}{\partial y_j^{(k)}}, \quad a_k \frac{\partial y_j^{(k)}}{\partial a_l} = -\frac{\partial H_l}{\partial x_j^{(k)}},$$

and

$$(4.44) \quad \frac{\partial \xi_j}{\partial a_l} = \frac{\partial H_l}{\partial \eta_j}, \quad \frac{\partial \eta_j}{\partial a_l} = -\frac{\partial H_l}{\partial \xi_j}.$$

Now equation (3.29) of Proposition 3.5 can be checked by direct computations as follows. Equations (3.25), (4.43), and (4.44) imply that  $\{H_l, H_\rho\}$  can be written explicitly as

$$(4.45) \quad \{H_l, H_\rho\} = \sum_{k=1}^{2m} \sum_{j=0}^M a_k \left( \frac{\partial x_j^{(k)}}{\partial a_l} \frac{\partial y_j^{(k)}}{\partial a_\rho} - \frac{\partial x_j^{(k)}}{\partial a_\rho} \frac{\partial y_j^{(k)}}{\partial a_l} \right) + \sum_{j=0}^M \left( \frac{\partial \xi_j}{\partial a_l} \frac{\partial \eta_j}{\partial a_\rho} - \frac{\partial \xi_j}{\partial a_\rho} \frac{\partial \eta_j}{\partial a_l} \right).$$

Then we use equations (3.13)-(3.20) to deduce that equation (3.29) indeed holds true. Thus Proposition 3.5 is proved. The equivalence of equations (3.33)-(3.35) to equations (3.13)-(3.20) can be checked by simple calculations. The equivalence of the formula for  $H_l$  in Proposition 3.6 and of equation (3.28) can be verified directly as well. Proposition 3.6 is proved.  $\square$

**4.5. Proof of Proposition 3.9.** Equations (3.44) -(3.49) is just a specialization of the general partial differential equations (3.13)-(3.20) in Proposition 3.3 to the case of the single interval  $(0, s)$ . Moreover, equation (4.42) implies

$$(sR_M(s))' = (-1)^{M+1} x_0(s) y_M(s), \quad R_M(s) := R_M(s, s),$$

or

$$R_M(t) = \frac{(-1)^{M+1}}{t} \int_0^t x_0(\tau) y_M(\tau) d\tau.$$

On the other hand, we have

$$\frac{d}{ds} \log \det(1 - K_M) = -R_M(s),$$

so

$$\det(1 - K_M) = \exp \left( (-1)^M \int_0^s \frac{1}{t} \left( \int_0^t x_0(\tau) y_M(\tau) d\tau \right) dt \right).$$

The formula for  $F_M(s)$  in the statement of Proposition 3.9 follows from the formula just written above by integration by parts. Proposition 3.9 is proved.  $\square$

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